

DEDEKIND, STRUCTURAL REASONING, AND MATHEMATICAL UNDERSTANDING

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1. Introduction

The last few decades have witnessed a broadening of the philosophy of mathematics, beyond narrowly foundational and metaphysical issues, and towards the inclusion of more general questions concerning “mathematical methodology” and “mathematical practice” (a development parallel to an earlier broadening of the philosophy of science). There is now widespread, and growing, interest in topics such as: concept formation and conceptual change in mathematics, the role of ambiguity and inconsistency in mathematical research, the applicability of mathematics, and even sociological or anthropological questions concerning the mathematical community. Part of this broadening, although a part that remains relatively close to foundational and metaphysical issues, is the turn towards a “new epistemology” for mathematics. The latter includes the study of topics such as: the role of visualization in mathematics, the use of computers in proving mathematical theorems, and the notion of explanation as applied to mathematics.^a

The present paper is a contribution to this new epistemology. More particularly, it is an attempt to bring into sharper focus, and to argue for the relevance of, two related themes: “structural reasoning” and “mathematical understanding”. As the notion of understanding is vague and slippery in general, as well as very loaded in philosophical discussions of the sciences, the latter label has to be handled with care, though. It will have to be clar-

^a Compare, e.g., Mancosu, Jørgensen and Pedersen,¹ Ferreirós and Gray,² Van Kerkhove and Van Bendegem,³ and Mancosu.⁴

ified what, if anything (or anything reasonably precise), is to be meant by “understanding” in connection with mathematics. Similarly, while talking about “structural” reasoning in mathematics may be suggestive, that term too requires further elaboration. My clarifications and elaborations will be tied to a specific historical figure and period: Richard Dedekind and his contributions to algebraic number theory in the nineteenth century. This is not an incidental choice; Dedekind’s case is particularly pertinent in this context, as I also hope to establish.

I will proceed as follows: In Section 2, I will provide a brief summary of Dedekind’s work on the foundations of mathematics, as well as of its usual perception in the philosophy of mathematics. In Section 3, I will turn to his more mainstream mathematical work, especially in algebraic number theory, including its usual perception by historians of mathematics. In the next few sections, the epistemological significance of this mathematical work will be explored further. In Section 4, I will review corresponding analyses in three pieces of secondary literature: Stein,⁵ Ferreirós,⁶ and Avigad.^{7, b} In Section 5, I will introduce the notions of style of reasoning and explanation to deepen their analyses. In Section 6, my views on mathematical explanation and, correspondingly, on mathematical understanding will be clarified further. Finally, in Section 7, I will indicate how the epistemological issues at the core of this paper can be seen as being of a piece with foundational and metaphysical issues.

2. Perceptions of Dedekind by philosophers of mathematics

While Dedekind did not publish any primarily philosophical writings, his foundational work is familiar to most contemporary philosophers of mathematics. His contributions in three areas, in particular, are well known: the foundations of analysis, the foundations of arithmetic, and the rise of modern set theory. Let me remind the reader briefly of those contributions, as well as of their typical characterizations by philosophers.

Dedekind is probably best known for his introduction and treatment of the real numbers in terms of “Dedekind cuts” (first presented in Dedekind⁸). This treatment is usually seen as a contribution to the “arithmetization of analysis” in the nineteenth century. In the twentieth century, it became part of the standard account of the real numbers within axiomatic set theory. The treatment is closely related to, indeed was based on, Dedekind’s anal-

^b For further details concerning these publications, see the bibliography. For references to other relevant literature, compare the following footnotes.

ysis of the notion of continuity (in the sense of line-completeness). That analysis was later codified as one of the axioms for a complete ordered field — one of the “Dedekind-Hilbert Axioms”, as they should perhaps be called — and is, as such, definitive for the classical conception of the real numbers.^a

Dedekind’s investigations into the foundations of arithmetic, in Dedekind,¹² are known almost as well. In that case he was, in effect, led to the “Peano Axioms” — or the “Dedekind-Peano Axioms” — for the natural numbers. He also proved what we now call the categoricity of this system of axioms; he constructed a standard model for it, in the form of a “simply infinite system”; and the whole account was grounded in an analysis of the methods of proof by mathematical induction and definition by recursion. Dedekind’s account of the natural numbers became, again, a standard part of set theory in the twentieth century, especially after it was made clear, by Zermelo, that it could be extended to ordinal numbers, induction, and recursion in the transfinite case.^b

As mentioning the notion of a simple infinity already flags, the approach taken in Dedekind¹² includes a systematic reflection on the notion of infinity, as well as on those of set and function. Especially important in this connection are: Dedekind’s explicit adoption of a general, extensional notion of set; his parallel adoption of a general, extensional notion of function (without reducing functions to sets); and his definition of infinity in terms of what is now called being Dedekind-infinite. Dedekind made other contributions to the early development of set theory as well, partly in correspondence with Cantor, such as his proof of the Cantor-Bernstein theorem.^c

Standard accounts of Dedekind’s foundational work, such as the one just given, lead naturally to three views about him: a) that he was a strong proponent, indeed one of the founding fathers, of “classical mathematics” (with his acceptance of the actual infinite, his adoption of generalized notions of set and function, his rejection of constructivist restrictions, etc.); b) that he was a main contributor to set theory, indeed to set theory con-

ceived of as a foundation for all, or at least large parts, of mathematics (with his set-theoretic treatments of the natural and real numbers, his analyses of continuity, induction, etc.); sometimes also, c) that he was as a direct precursor of, and a strong influence on, the “formal axiomatic” approach championed by Hilbert and Bernays later (with his implicit formulation of the axiom systems for the natural and real numbers, his attention to questions about categoricity, consistency, etc.).^d

There is a lot of truth in these standard views about Dedekind, and they do bring out important aspects of his work. In what follows I will attempt to show, however, that in some ways they do not go far enough — they neglect or underemphasize a philosophically significant dimension of Dedekind’s work. To prepare the corresponding arguments, it helps to turn to his less foundational and more straightforwardly mathematical works, starting with their standard perceptions by historians of mathematics.

3. Perceptions of Dedekind in the history of mathematics

In addition to his foundational work, Dedekind made several well-known contributions to other parts of mathematics, especially to algebra and related fields. For instance, his work contains one of the first, probably even the first, modern presentation of Galois theory. He also published important papers on what were later called “lattices”; in fact, he characterized lattices for the first time in an explicit, general, and conceptually clear manner. And in collaboration with Heinrich Weber, he introduced a novel approach to the study of algebraic functions, connected with a new treatment of Riemann surfaces, and leading to a purely algebraic proof of the celebrated Riemann-Roch theorem.

Dedekind’s most important and most influential mathematical work, however, was in algebraic number theory. In historical accounts of nineteenth-century mathematics he is, thus, routinely mentioned for two things: his invention of the theory of ideals, seen as a crucial new tool in the study of algebraic integers and algebraic number fields; and his introduction, in that context, of the abstract mathematical notion of a field, applied by him to all subfields of the complex numbers and including adopting the word “field” for this purpose (or rather the corresponding German word,

^a For Dedekind’s role in the arithmetization of analysis, see Boyer and Merzbach⁹ (ch. 25, pp. 563–66), and Cooke.¹⁰ For more on the “Dedekind-Hilbert Axioms”, see Awodey and Reck.¹¹

^b For Dedekind’s contributions to the foundations of arithmetic, see Reck¹³ and Ferreirós.¹⁴

^c For a historically and philosophically rich discussion of Dedekind’s role in the development of set theory, see Ferreirós,⁶ especially chs. 3, 4, and 7.

^d Seeing Dedekind as a proponent of classical mathematics is standard wisdom, I believe. For Dedekind’s role in the development of modern set theory, see again Ferreirós.⁶ For a sophisticated interpretation of Dedekind as a precursor of Hilbert and Bernays, see Sieg and Schlimm.¹⁵

“Körper”).^a Both contributions occur in supplements to lectures notes on number theory, based on lectures by his teacher Dirichlet but edited by Dedekind, and most maturely, in the fourth edition of that work (Dirichlet and Dedekind;¹⁷ compare also Dedekind¹⁸).

Two general aspects of this mathematical work and of its impact are typically emphasized in historical accounts. First, Dedekind’s novel approach to algebraic number theory, as embodied in his ideal theory, did not go unchallenged and unopposed. Kronecker’s parallel work in this area, culminating in his divisor theory, was seen as a significantly different alternative to Dedekind’s from early on. Kronecker himself kept emphasizing the more concrete, finitary, and constructive aspects of his theory, while being critical of the abstract, infinitary, and non-constructive aspects of Dedekind’s. Second and in spite of such criticisms, Dedekind’s approach had a strong influence on twentieth-century mathematics, through the works of Hilbert, Noether, van der Waerden, Bourbaki, and others. This influence was often acknowledged explicitly, e.g. by Emmy Noether. Reflecting on the basic methodological orientation of her own, itself very influential, work in algebra and topology, she stated: “It’s all already in Dedekind!”^b

As Dedekind’s mathematical works tend to be much less well known to philosophers than his foundational contributions, let me add a bit to this brief summary (before proceeding to a deeper analysis in later sections). In particular, what are the main goals and challenges in Dedekind’s and Kronecker’s theories in algebraic number theory?

Both theories were heavily indebted to earlier works by Gauss, Dirichlet, and Kummer. For all of these mathematicians the basic goal was the solution of various algebraic equations. A famous example is provided by Fermat’s Last Theorem, which concerns the existence, or lack of existence, of integer solutions to the equation $x^n + y^n = z^n$, for various exponents n . Gauss and Kummer approached this (very difficult) issue by studying certain extensions of the ordinary integers, as well as of the fields of numbers that contain them. Gauss considered what happens when you add the “Gaussian integers” ($a + bi$, with a and b regular integers and $i = \sqrt{-1}$); Kummer investigated more complex “cyclotomic integers”. Along the way, it became clear that a crucial issue, and a major stumbling block, was the following: In some such extensions of the ordinary integers — in some “in-

^a See again Boyer and Merzbach⁹ (now ch. 26, pp. 594–6), as well as Stillwell¹⁶ (ch. 21).

^b See McLarty¹⁹ for the source of the quotation (p. 188), and more generally, for Dedekind’s influence on Noether. Edwards²⁰ provides a comparative discussion of Dedekind and Kronecker.

tegral domains”, as Dedekind called them^c — the familiar theorem about unique factorization into powers of primes fails. A crucial question became, then, whether a suitable alternative for such factorization could be found.

Kummer attempted to recover unique factorization by introducing “ideal numbers”. While this led to striking progress, some basic questions remained. In particular, how exactly was one to think about the nature of these new “numbers”; and what was the best way to generalize Kummer’s approach, if this was possible at all? As a consequence, the range of applicability of his ideas remained unclear, to some degree even the validity of his results. Both Kronecker and Dedekind tried to justify and extend Kummer’s work. Kronecker did so by considering in depth — and as part of an essentially computational task (starting from a finitary basis and preserving decidability) — a range of constructible domain extensions. Dedekind investigated — in a more general, abstract, and non-constructive way — arbitrary algebraic number fields and the integral domains they contain. He also replaced Kummer’s “ideal numbers” by his “ideals”, defined in an explicitly set-theoretic way (as certain infinite sub-sets of the complex numbers), and he recovered unique factorization that way.^d Both Kronecker’s and Dedekind’s approaches led to further results right away, as well as to important developments later on.^e

Kronecker’s and Dedekind’s works are similar insofar as both constitute “arithmetical” approaches to algebra. (They are two instances of the “arithmetization of algebra” in the nineteenth century, parallel to the more familiar “arithmetization of analysis”.) Apart from that, they differ markedly. Comparing the two mathematicians and their lasting impacts, the historian of mathematics Harold G. Edwards comments:

Kronecker’s brilliance cannot be doubted. Had he had a tenth of Dedekind’s ability to formulate and express his ideas clearly, his contributions to mathematics might have been even greater than Dedekind’s. As it is, however, his brilliance, for the most part, died with him. Dedekind’s legacy, on the other hand, consisted not only of important theorems, examples, and concepts, but of a whole

^c In current terminology, an integral domain is a ring that is commutative under multiplication, has a unit element, and has no divisors of zero.

^d An ideal I in an integral domain (or, more generally, a ring) R is a subset that forms an additive group such that, for all $x \in I$ and $y \in R$, $xy \in I$. The crucial theorem is: In a domain R of algebraic integers, any ideal I of R can be represented uniquely (except for the order of the factors) as a product of prime ideals.

^e Besides Edwards²⁰ and McLarty,¹⁹ compare Reed²¹ (ch. 4) and Corfield²² (ch. 8).

style of mathematics that has been an inspiration to each successive generation. (Edwards,²⁰ p. 20)

On the surface, this passage is complimentary of Dedekind's work, highlighting his "great contributions" and his corresponding "legacy". However, the way in which Kronecker's "brilliance" is juxtaposed to Dedekind's "ability to formulate ideas clearly" may give one pause. Kronecker certainly was a brilliant mathematician; and Dedekind had that ability. But is the latter all that is noteworthy about Dedekind in this connection; doesn't his work exemplify other, equally or more significant, virtues as well?^f

Edwards also attributes "important theorems, examples, and concepts" to Dedekind. More intriguingly, he mentions a Dedekindian "style of mathematics" that inspired later generations of mathematicians. The latter raises another question, however: How is the word "style" to be understood here; in particular, is it used in a merely psychological or sociological sense, or is more at issue?^g Raising this question is also meant to lead us beyond Edwards' remark. The further, more important issue, for present purposes, is whether "style" could be used in a philosophically more substantive sense in this context. Or more generally, is there anything else to be said about the epistemological significance of Dedekind's approach, compared to Kronecker's and in itself? These are the questions I want to turn to now. Actually, a few other philosophers of mathematics have already started to move in that direction, and I want to follow their lead.

4. Philosophical analyses of Dedekind's mathematical work

There is a relatively small, but illuminating and suggestive, series of commentaries in the literature in which Dedekind's approach to algebraic number theory, and with it, his methodology in general, is analyzed with an eye towards its epistemological significance. It would be worth reviewing, and then building on, all of them; but I will have to restrict myself to three

^f In the background of such remarks are Edwards' strong and well-known sympathies for a Kroneckerian approach to mathematics; compare Edwards,^{23,24} To be fair, he does have more to say about Dedekind, also in Edwards;²⁵ but the general perspective assumed is always Kronecker's.

^g Other historians of mathematics, such as Ivor Grattan-Guinness, have written about the "fashion" Dedekind's work inspired, as well as the "popularity" set theory gained later on (Grattan-Guinness,²⁶ p. 535). Elsewhere Edwards writes about a "new orthodoxy" in this connection, one that was "consolidated by Hilbert" and that "has reigned ever since" (Edwards,²⁴ p. ix).

here: Stein,⁵ Ferreirós,⁶ and Avigad.^{7,a}

The focus in Stein's paper is not so much on Dedekind, but on a more general change that took place in nineteenth-century mathematics, especially in its methodology. Stein associates this change with a group of mathematicians who influenced each other directly: from Gauss and Dirichlet through Riemann, Cantor, and Dedekind, on to Hilbert and Minkowski. Dirichlet is seen as the "poet's poet" in this group. Dedekind, although not the primary focus of Stein's study, is very important as well, especially because of his close association with Dirichlet. In a retrospective tribute to Dirichlet at the beginning of the twentieth century, Minkowski captured the transformation at issue in a nutshell; in Minkowski's memorable phrase, it consisted of adopting Dirichlet's guiding principle "to conquer the problems with a minimum amount of blind calculation, a maximum of clear-seeing thought" (quoted in Stein,⁵ p. 241).

According to Stein, this principle shapes Dedekind's approach quite generally. It guides his reconceptualization of the foundations of arithmetic and analysis in terms of abstract sets and functions, instead of concrete numerals, formulas, and their intuitive applications, as was usual before; it underlies his innovative presentation of Galois theory in terms of field extensions and their automorphisms, rather than substitutions in formulas and functions; and it is perhaps most evident in his algebraic number theory. Concerning the latter, the contrast between "clear-seeing thought" and "blind calculation" amounts to this: While Kronecker, like Kummer and Gauss before him, works with a restricted number of constructible cases, trying to extract computational information, Dedekind considers an enlarged class of algebraic structures, here number fields, always searching for general, not necessarily decidable, and characteristic concepts.

Quoting related remarks by Cantor, Stein characterizes the shift towards the more "conceptual" mathematics to be found in Dirichlet, Dedekind, Riemann, and others also as a "freeing" of mathematics. Previous mathematics tended to be confined to the physically applicable and intuitive, especially the geometric, on the one hand, and to the calculable and constructive, on the other. It is not that such a narrower conception — with its emphasis on "the applied force of the formula", to quote Minkowski again — can't lead to new results. Indeed, Kronecker's brilliant extension of Kummer's work

^a Also highly relevant are: Haubrich,²⁷ Reed,²¹ and Corry.²⁸ I will focus on Stein, Ferreirós, and Avigad because their expressly philosophical concerns seem closest to mine. But the difference is only gradual, and I intend to pay close attention to the others in future work.

shows that it can. But the reorientation advocated by the champions of “mathematical freedom” can, and did, lead to an important broadening of mathematics, including the formation of various new and fruitful concepts. To use one of Stein’s own happy phrases, it leads, indeed it gives pride of place, to the “free exploration of conceptual possibilities”.^b

As Stein notes, Dedekind’s allegiance to such a more freely exploratory mathematics went hand in hand with, indeed involved centrally, the use of set-theoretic techniques and proofs, instead of the earlier reliance on intuitive constructions and calculations. Stein elaborates on that aspect to some degree. In Ferreirós⁶ a lot more can be found on the topic. Ferreirós’ book does again not focus on Dedekind alone. It discusses his employment of set-theoretic techniques in the context of a more general account of the rise of modern set theory, from the nineteenth into the twentieth century. According to Ferreirós’ account, Dedekind is one of the central figures in the early history of set theory. This is so, among others, because he explicitly adopted two ideas: to treat sets as mathematical objects in themselves; and to allow for the use of infinite sets, indeed to use them as a central tool for concept formation in mathematics.

Besides shedding new light on Dedekind’s role in the rise of modern set theory, Ferreirós’s discussion of him also confirms, and extends, some of Stein’s insights into the significance of his work in algebraic number theory, and of his mathematical methodology more generally. As analyzed by Ferreirós, this methodology involves: the consideration of whole classes of systems, e.g., of the class of arbitrary sub-fields of the complex numbers; their abstract treatment in terms of general laws, such as the laws that characterize number fields, integral domains, or simple infinities; and more specifically, the definition of operations on mathematical systems in terms of their behavior as sets, thus independently of particular formalisms and calculations based on them.

The last point — the preference for general, abstract, and representation-invariant specification of mathematical operations and objects — can be exemplified well by a particular aspect of Dedekind’s theory of ideals. Dedekind labored for quite a while — through various supplements to Dirichlet’s “Lectures on Number Theory”, published over three decades — to find a good, perhaps even the best, way to define an ex-

tended notion of “integer”, applicable to number fields in general; similarly for “ideal divisor” and, especially, for “prime divisor” (as needed to ensure unique factorization). Now, Kronecker worked on solving a parallel problem. But characteristically, Kronecker’s solution was not meant to apply as generally; it did not employ set-theoretic techniques, especially not the use of infinite sets; and it was tied to specific formalisms and representations (as needed by Kronecker to ensure computability). In the end, their respective solutions had different advantages and disadvantages.

In the present paper, I am mainly concerned with the philosophically significant advantages of Dedekind’s approach, in this case and more generally. It is standard to assume that Dedekind constructed his ideals and related mathematical objects in an explicitly set-theoretic way because the “ideal numbers” appealed to by Kummer had provoked mistrust or doubt, since they lacked an explicit, secure foundation. But a standard rejoinder, especially by constructivists, is this: As the “foundational crisis” in the early twentieth century has shown, the use of sets, especially infinite sets, is not necessarily more secure; but then, their use should be seen with mistrust too, shouldn’t it? Whether or not one agrees with this rejoinder, I do not think — and this should have become apparent by now — that providing a secure foundation for certain parts of mathematics was the only objective for Dedekind. Arguably it was not even his main objective, particularly in algebraic number theory; nor was it his philosophically most significant achievement, at least from a methodological point of view.

An additional, more recent attempt to get at Dedekind’s main methodological achievements — focused squarely on his approach to algebraic number theory — is Avigad.⁷ In this paper, several of Stein’s and Ferreirós’ observations are confirmed yet again, while many number-theoretic details are added and the analysis of their significance is deepened. Like Stein and Ferreirós, Avigad mentions the use of set-theoretic techniques by Dedekind, including his acceptance of the actual infinite. Once more, he emphasizes the contrast between Dedekind’s abstract, conceptual, or structural approach, on the one hand, and Kronecker’s focus on algorithmic tractability and decidability, on the other. And once again, he puts his finger on Dedekind’s aim to find general, mathematically fruitful concepts or characteristics.

A related aspect, mentioned already in connection with Ferreirós, is discussed in considerable detail by Avigad as well: the fact that, according to Dedekind, mathematical objects and operations should be defined in a representation-invariant way. Avigad also sheds further light on two related ways, touched on by Stein, in which Dedekind characterizes his own pro-

^b In Tait²⁹ the emphasis on “free mathematics” is discussed further, in connection with Cantor and Dedekind; compare also Reck.¹³ The search for “characteristic concepts”, by Riemann, Dedekind, and Frege, is discussed more in Tappenden,^{30,31} and the relevant parts of Mancosu.⁴

cedure: as an attempt to distinguish “internal” from “external” properties of mathematical entities; and as an attempt to keep mathematical theories as “pure” as possible, i.e., to avoid the admixture of “foreign” elements. Examples are: Dedekind’s resistance to taking particular formalisms, typically tied to individual cases, to be more important than the underlying mathematical structure; and his rejection of appeals to geometric ideas in his treatment of the foundations of arithmetic and analysis.

At several points, Avigad summarizes what is crucial about Dedekind’s approach, and what distinguishes it most significantly from Kronecker’s, in terms of the adoption of certain methodological values and principles. Central among them are: the preference for conceptual reasoning, as opposed to algorithmic calculation; the preference for abstraction, generalization, and the uniform treatment of as large a number of cases as possible, as opposed to gaining computational information by focusing on a restricted number of constructible instances; and the preference for identifying characteristic internal properties, thus for a certain kind of simplicity and purity, in the sense just indicated. It should be emphasized that these values are in addition to, and that they complement, Dedekind’s use of set-theoretic and infinitary tools and techniques. Perhaps they are even what motivates and what ultimately justifies his adoption of them?

I hope it is plausible by now that Dedekind’s work does not just stand out because of his exceptional expository ability.^c Nor is it entirely satisfactory, in the end, to characterize the difference between his approach and Kronecker’s simply by pointing to Dedekind’s adoption of set-theoretic and infinitary techniques, in contrast to Kronecker’s constructivist and finitist procedures, although that is again part of the story. Various additional, perhaps more basic, and arguably more important features have come to the fore, in Stein’s, Ferreirós’, and Avigad’s studies. Now, can these additional features be analyzed in any further, deeper way? More specifically again, can the sense in which they are significant epistemologically be brought out more sharply, after having identified them? I will attempt to do so in the next two sections.

^c To prevent a possible misunderstanding, I do not mean to discount this aspect completely. Being able to present ideas clearly is crucial, e.g., in a pedagogical context; and some of Dedekind’s most important ideas arose in just such a context; see Dedekind,⁸ p. 1. But there is quite a bit more to be said about Dedekind’s approach, especially from an epistemological point of view.

5. Styles of reasoning and mathematical explanation

At this point, I want to come back to Edwards’ remark about a Dedekindian “style of mathematics”. One question I raised earlier was whether “style” is used here in a merely psychological or sociological sense, or whether more is at issue. More importantly, could “style” be employed in a philosophically more substantive sense in this connection, whether or not Edwards does? Let us consider the latter issue in some detail now.

In contrast to using “style” in a psychological, sociological, or anthropological sense (including the idea of “national style” in mathematics), also in contrast to using it in a personal, aesthetic, or art-historical sense (the “style” of a writer or painter), there is the way in which the notion has been employed, and codified, by Ian Hacking.^{33,34} What Hacking talks about specifically, in the context of the history and philosophy of science, are “styles of reasoning”.^a While he is reluctant to define what a style in his sense is, at least in any reductive or formulaic way, he provides various examples, including: the postulational style that characterized the mathematical sciences in Ancient Greece; the experimental style that arose in early modern science; and the statistical and probabilistic style that, in the nineteenth century, began to shape the social sciences. And he elaborates on what is significant about such styles, namely:

Every style of reasoning introduces a great many novelties, including new types of: objects; evidence; sentences, new ways of being a candidate for truth and falsehood; laws, or at any rate modalities; possibilities. One should also notice, on occasion, new types of classification and new types of explanation (Hacking,³⁴ p. 189).

As one may also put it, Hacking is talking about different kinds of “cognitive style”.

An aspect that makes the notion of cognitive style, in Hacking’s sense, useful is that it foregrounds philosophical issues, including epistemological issues (also related metaphysical ones). Thus, what matters are “ways of being a candidate for truth and falsity”; equally crucial are new types of “evidence”, “laws”, “classification”, and “explanation”. Along such lines, the focus is on general, and often novel, ways in which scientists conceptual-

^a Hacking’s discussion of “style” builds on A. C. Crombie’s historical work on “styles of scientific thinking”; see Hacking.³⁴ In connection with mathematics, similar uses occur in several of the chapters of Mancosu, Jørgensen and Pedersen,¹ e.g., Høyrup.³⁵ Compare also Mancosu.³²

ize issues, formulate problems, evaluate solutions, systematize results, etc. Basically, a style of reasoning is a distinctive, integrated manner of doing those kinds of things.^b Understood as such, a reasoning style can crystallize in the work of one or more thinkers, so as then to shape the direction of a discipline for a while.

Most of the examples of reasoning styles considered by Hacking come from the natural and social sciences. They are also very broad and general. He mentions only a few mathematical examples, such as the postulational style of the mathematical sciences in Ancient Greece. But nothing rules out the application of this notion to other cases in mathematics, as one may argue, including more recent ones. In fact, what we considered above — the distinctively “conceptual” or “structural” approach to mathematics championed by Dedekind — appears to be a very good example.^c It involves all the features highlighted by Hacking: new types of evidence (definitions and constructions involving set-theoretic techniques, including uses of the actual infinite, non-constructive and non-computational proofs, etc.); new laws (laws for generalized classes of structures, appeals to the general notions of set, function, etc.); new types of classification (simple infinities, number fields, groups, lattices, etc.); and new types of explanation (based on characteristic concepts, on novel ways of relating phenomena, etc.). To have a slogan, we may talk about a “structural style of reasoning” as exemplified by Dedekind’s work.^d

A general way in which talking about a style of reasoning in this connection is helpful is by drawing attention to the epistemological dimension of Dedekind’s works. In addition, the specifics of Hacking’s proposal — concerning the introduction of novel kinds of evidence, law, classification, explanation, etc. — provide us with conceptual tools for deepening the analysis. It would be interesting to consider, in detail, each of these tools

^b For a related account of what is important epistemologically in this context, as well as a comparison to how the notion of “style” is used in art history, see Davidson.³⁶ As Davidson emphasizes, “styles of reasoning give systematic structure and identity to our thought” (p. 141).

^c Alternatively, one could try to analyze Dedekind’s approach as an example of a Kuhnian “paradigm” (both in the sense of “exemplar” and “disciplinary matrix”) or as an example of a Lakatosian “research programme”. Ferreirós⁶ contains remarks about Dedekind along Kuhnian lines, while Corfield²² pursues a Lakatosian direction. In what follows, I will indicate what is particularly helpful about the notion of style or reasoning in our context. (More on “research programmes” in footnote d, Section 6.)

^d Dedekind’s approach and position have also been called “logicist”, e.g. by his contemporaries C. S. Peirce and Ernst Schröder; compare the corresponding discussion in Ferreirós.⁶

and what can be done with it. Let me single out the last one here: the notion of explanation. I think it is correct, and to the point, to see Dedekind’s structural style of reasoning as involving a new type of mathematical explanation. However, it then needs to be clarified what that implies. Also, how does talking about explanation in the context of mathematics relate to discussions of that notion in the philosophy of science, if at all?

In general philosophy of science, the notion of explanation is often discussed in connection with the notion of causation.^e Along such lines, what philosophers of science — philosophers of physics, biology, sociology, economics, etc. — are interested in is to get at the sense, and the precise forms, in which appealing to the cause of an event or phenomenon is, or can be, explanatory. Now, it may seem that causation has no role to play in mathematics, which may also make talk about mathematical explanation seem dubious. In one sense this is surely correct, namely if “cause” is used in the narrow sense of “efficient cause” (analyzed in terms of natural laws, capacities, counterfactual dependence, etc.). Then again, one can use “cause” in a more general sense as well. In that sense, anything that is given as the answer to a why-question counts, especially if the answer takes the form of “because...”. In mathematics we can, and sometimes do, ask why-questions. As an example, consider: “Why are certain kinds of algebraic equations solvable by integers while others aren’t?” Answers to such questions may then be taken to provide “explanations”, perhaps even “causal explanations”.^f One might also want to compare the corresponding explanatory power of mathematical theories.

This last remark calls for further clarification, or for a distinction that will be helpful. The distinction is between a “local” and a “global” sense of “explanatoriness”, both in the sciences and in mathematics. In the case of mathematics, the local sense concerns the manner in which a particular proof of a theorem can be seen as explanatory, or as more or less explanatory than other proofs of the same result. (The literature in philosophy of mathematics contains some proposals for how to think about being explanatory in this local sense, e.g., in Steiner;⁴⁰ more on this in the next section.) The global sense of explanatoriness, in contrast, concerns the

^e Not always; Hempel’s and Kitcher’s discussions of explanation are exceptions. But compare Salmon³⁷ and subsequent works by Cartwright, Humphries, Salmon, and Woodward, among others.

^f To be more careful, one should distinguish between explanatory and other why-questions in this context. In making answers to why-questions central to the issue of explanation, I follow van Fraassen³⁸ and the erotetic literature on which he relies. In addition, I am heavily indebted to Wright.³⁹

way in which a whole theory or a general approach to a subject matter is explanatory, or can be evaluated as more or less explanatory than other theories or approaches. (Here too the literature contains proposals, e.g., in Kitcher;⁴¹ again, more on this below.)

With these clarifications and distinctions in place, I can now state a general claim. The claim is that an important aspect, perhaps even the main aspect, of what makes Dedekind's structural style of reasoning significant epistemologically is its characteristic explanatory power in the global sense. A lot more would have to be said to substantiate and defend this claim fully. I will only have space for a few additional remarks in the next section, before wrapping things up more generally.

6. Explanations, background assumptions, and understanding

While it will be crucial for us to get clear about the explanatory power of Dedekind's approach in the global sense, as just suggested, it helps to start with the local sense, especially since the two senses are not unrelated. In particular, it helps to examine a specific proposal for how to think about explanatoriness at the local level.

The proposal I have in mind is due to Mark Steiner; in his own words:

[A]n explanatory proof makes reference to a characteristic property of an entity or structure mentioned in the theorem. It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses; more, we should be able to see as we vary the object how the theorem changes in response. (Steiner,⁴⁰ p. 143)

It is not obvious whether this criterion for being an explanatory proof applies generally, nor whether it characterizes being explanatory fully. It has been criticized seriously in connection with other examples.^a Nevertheless, Steiner's criterion looks promising in connection with Dedekind's work, especially in algebraic number theory. As we saw, it was a main goal for Dedekind to come up with "the right concepts", and thus to identify "characteristic properties" of various entities and operations. Moreover, the right concepts for him were exactly those that apply to a wide range of cases, or even, allow us to distinguish those cases for which a certain proof worked from others.

^a For references and further discussion, see Hafner and Mancosu⁴² and Mancosu.⁴

Let me connect this point with an insight gained by looking at explanations as answers to why-questions. A main advantage of approaching the notion of explanation that way — besides the fact that it makes its applicability in mathematics plausible — is that we are directly led to the crucial role of background assumptions. Consider asking a why-question, in the form "why p?", and answering it, with "because q". Evidently, the whole exchange can only be successful if a number of presuppositions are in place. Two of these presuppositions are especially noteworthy. The first is the availability and determinate nature of what is often called the "contrast class" for p.^b It has to be clear, that is, what the alternatives to p are in the context at hand: p as opposed to p', p", etc.; otherwise it is not even clear what question is being asked by using the phrase "why p?" Second, it has to be clear, again in context, what kinds of explanatory factors or "causes" are relevant.^c These two presuppositions are closely related. Distinguishing p from p', p", etc. will have to be in terms of specific features; and those features will be identical with, or intimately related to, the explanatory factors relevant in the context at hand.

For illustration, consider again our mathematical example from above: "Why are certain kinds of algebraic equations solvable by integers while others aren't?" In formulating the question in this way, the appeal to a contrast class is readily apparent, at least in a general way. The fact that certain explanatory factors are presupposed is more hidden. Now, compare Dedekind's approach again with Kronecker's. For Kronecker, the contrast class consists of a tightly circumscribed range of equations, corresponding to number fields constructed finitistically; and the presupposed explanatory factors are computational ones. For Dedekind, the contrast class is determined by an enlarged class of number fields, thus consisting of a larger number of equations; and the relevant explanatory factors involve entities defined set-theoretically and considered structurally. Altogether, the most radical differences between Dedekind's and Kronecker's approaches can be located at this level, I would suggest. They consist of differences in the general background assumptions for their respective explanatory enterprises.

We are now also in a position to relate the global and local senses of explanatoriness to each other, at least in our case. Consider again Steiner's

^b Van Fraassen³⁸ contains an illuminating discussion of the notion of contrast class.

^c In Wright,³⁹ the author talks about a presupposed "causal matrix" in this connection. I should add that, with respect to this second aspect, Wright's and my approach differs significantly from van Fraassen's (which appeals to a somewhat mysterious, and often criticized, "relevance relation" at this point).

criterion: that at the local level, i.e., the level of particular theorems and proofs, what is crucial is to identify the “characteristic property of an entity or structure”; also, to show that “if we substitute in the proof a different object of the same domain, the theorem collapses”; or even, to establish how “as we vary the object the theorem changes in response”. Notice that such a criterion only has a chance of applying if it is clear, first, what the relevant “objects of the domain” are and, second, which kinds of “characteristic properties” count. And again, those are exactly the two ways in which Kronecker’s and Dedekind’s approaches differ radically. We get the following consequence: Considered just locally, the two approaches are very hard to compare, if not incommensurable, since they differ so much in their respective background assumptions (thus the continuing disagreement between Dedekindians and Kroneckerians). At the global level, there may be more room for comparative evaluation. In particular, we can ask in which way, and to what degree, various background assumptions have proved fruitful mathematically.^d

A comparative assessment at the global level will still not be straightforward, partly because fruitfulness is hard to quantify, partly because it is relative to the goals one is pursuing (e.g., computational versus structural goals), and partly because it becomes manifest only over time. It may also go through varying phases; e.g., while an approach may not be fruitful for a while, it may pick up again later.^e And in any case, assessing degrees of fruitfulness is not the same as providing criteria for explanatoriness, neither at the local level (where something like “getting at the core of things” seems to play a role, together with values such as simplicity and purity, in ways that are hard to capture^f) nor at the global level (where unification, systematicity, etc. seem to play some role, although it is not clear

^d Here Lakatos’ suggestions for how to evaluate “research programmes” may be of help. Compare again Corfield²² (and footnote c, Section 5), earlier also Hallett,⁴³ among others. As to incommensurability, I do not mean to push this issue too far. It may be possible to find background assumptions and a framework within which comparisons can be made, although the difficulty will be finding ones acceptable to both sides.

^e Kronecker’s approach was quite fruitful initially; it was then overshadowed, for some time, by work along Dedekindian lines; but it became fruitful again in the middle of the twentieth-century, in research by Grothendieck and others. Compare again Reed²¹ (ch. 4) and Corfield²² (ch. 8).

^f Explanatory power at the local level, like mathematical understanding in general (see below), may be too vague or multi-faceted a notion to be captured in any simple formula. However, there is some recent research in automated theorem proving, as discussed in Avigad⁴⁴ and Vervloesem,⁴⁵ which contains potentially fruitful, and very application-oriented, reflections on related issues.

which one exactly^g). Still, I hope that some new light has been shed on how to analyze the differences between Dedekind’s and Kronecker’s methodologies more deeply, and especially, on what makes Dedekind’s distinctive and noteworthy.

Let me add two further clarifications, now of the sense in which this all concerns epistemology. Both clarifications will involve the notion of understanding, which first needs to be clarified in itself. For present purposes, I want to use “understanding” not in contrast to “explanation” (as was common in nineteenth- and early-twentieth-century debates about the unity of science, i.e., about whether the methodologies of the human and the natural sciences are fundamentally different or not). Rather, “understanding” and “explanation” are taken to be correlative terms, along the following lines: What a successful explanation does is to improve our understanding of things; and an explanation is better the more it does so.^h The main claim I argued for above can then be put thus: The most characteristic, and perhaps the most valuable, aspect of Dedekind’s approach is the specific way in which it allows us to understand mathematical phenomena.ⁱ

Second, epistemology is often understood to be the philosophical study of human knowledge, and specifically, of its forms of justification and its connections to truth, also its means, conditions, limits, etc. However, for our purposes this is too narrow — it tends to exclude the topic of understanding from epistemology. Insofar as that is the case, “understanding” stands opposed to “knowledge”. In a recent paper, Howard Stein makes a related point, by distinguishing between the “enterprise of understanding” and the “enterprise of knowledge” (Stein,⁴⁷ p. 135). Both enterprises are important, as he emphasizes; they are also often intricately intertwined. Nevertheless, one can distinguish them conceptually. The enterprise of knowledge concerns what epistemologists typically focus on: justification, truth, etc. The enterprise of understanding, in contrast, has to do with our “grasp of ideas or concepts” and their “clarification” (ibid.). I would add that much of what we discussed above — distinguishing different styles of reasoning (based on different sets of concepts), considering their explanatory power (partly in

^g Compare Kitcher⁴¹ and the ensuing debate. Very briefly, I doubt that either unification or systematicity, in themselves, account for explanatoriness, especially at the local level; but they seem to play some role, perhaps of a supplementary kind, in connection with global explanatory power.

^h Here I again follow Wright,³⁹ who in turn builds on Scriven, Austin, and Wittgenstein.
ⁱ Related discussions of mathematical understanding, including its connection to proof, can be found in Tappenden,⁴⁶ Avigad,⁴⁴ Vervloesem,⁴⁵ and the corresponding parts of Mancosu.⁴

terms of “finding the right concepts”), etc. — falls within the latter as well.

Put in these terms, another main goal in this paper has been to make evident that Dedekind’s approach to mathematics is worth studying as part of the “enterprise of understanding”, and thus as part of epistemology understood in a broad sense.

7. Connections to foundational and metaphysical issues

My focus in this paper has been on epistemologically significant aspects of Dedekind’s work, including clarifying the sense of epistemology involved. A striking feature of his approach is, however, that it forms a tightly integrated whole. More specifically, epistemological aspects are tied closely to foundational and metaphysical aspects.

What I have in mind here is the following: As argued above, Dedekind’s characteristic way of understanding mathematical phenomena — in terms of his abstract, conceptual, or structural style of reasoning and explanation — involves corresponding background assumptions. Among them are assumptions about the factors one can appeal to in definitions, constructions, and proofs. For Dedekind, unlike for Kronecker, infinite sets, a generalized notion of function, etc., are available; thus, he gives definitions of various operations in terms of their set- and function-theoretic behaviors, independently of particular forms of representation and methods of calculation. With respect to the contrast classes assumed, we noted his use of enlarged classes of objects and structures, the preference for finding uniform treatments for them, etc. Overall, mathematical phenomena are treated in abstract relational and functional, thus structural, terms.

These aspects are crucial for Dedekind’s work in algebraic number theory, as I argued above. But not only that; the same aspects are also characteristic for his other mathematical works: his contributions to Galois theory, to the theory of algebraic functions, to lattice theory, etc. In fact, they even shape Dedekind’s foundational works, including his treatments of the natural and real numbers. Here too, we find the use of set-theoretic constructions, the acceptance of the actual infinite, the employment of a general notion of function, the consideration of generalized classes of cases, their treatment in abstract relational and functional terms, the search for internal, characteristic properties, etc. In other words, the same conceptual tools are employed throughout.

It is tempting to think, again, that it is Dedekind’s recognition of their fruitfulness in his mathematical work — his realization of how instrumental they are in increasing our understanding of, say, the solubility of algebraic

equations — that underwrites the use of these tools also in his foundational work. More likely, perhaps, is that he realized their explanatory power in both cases together, so that there was mutual reinforcement.^a In either case, we can see that mathematical and foundational concerns need not be as separate as is often assumed. If I am right, they are of a piece in Dedekind’s work.^b

Finally, the same point can be made about the close connection between epistemological and metaphysical aspects in Dedekind’s work. A central feature of his methodology is to study mathematical objects and operations not in terms of particular formalisms or symbolic representations. Dedekind recognized that it is epistemologically fruitful — that it increases our understanding in mathematics — if we investigate them, instead, in set-theoretic, abstract relational, and generalized functional terms. But making this shift also leads away from conceiving of the nature of mathematical entities and phenomena in two traditional ways: along narrowly formalist lines, so that all we are dealing with are empty symbols, mere formulas, etc.; in broadly physicalist terms, i.e., by making empirical applications of mathematics essential, so that numbers, e.g., are conceived of in terms of concrete quantities. In other words, Dedekind’s epistemological shift calls into question formalist, physicalist, and similar metaphysical views.^c

What Dedekind’s methodology suggests, instead, is to think of mathematical objects, concepts, and functions in structuralist terms. The resulting metaphysical position — Dedekind’s “logical structuralism” — has already been analyzed in Reck,¹³ but without paying much attention to the epistemological side, as elaborated in the present paper. In the end, metaphysical and epistemological aspects are flip sides of the same coin, in

^a To be more definite here, the precise chronology of Dedekind’s main ideas and results would have to be established (by studying his *Nachlass*, correspondence, etc.). I intend to do so in future work.

^b In Tappenden,³¹ the same point is made about Riemann and Frege. The widespread separation of mathematical and foundational concerns, by philosophers and mathematicians, is illustrated in it as well.

^c Whether or not formalist, physicalist, and related metaphysical positions should be rejected completely, and Dedekind’s simply adopted, is another question. His “logical structuralist” position is not without its weaknesses; other alternatives have come up since Dedekind’s time; and even a position such as narrow formalism has led to important insights, as Kronecker’s case illustrates. Moreover, there may not be one metaphysical position that does justice to mathematical facts and phenomena in all their richness, especially since the practice of mathematics keeps evolving. Still, Dedekind’s approach has led to novel and deep insights as well, as seems indubitable, at least for those who do not reject its background assumptions.

Dedekind's case and more generally. If this is correct, only a joint treatment can do full justice to either side. With respect to this conclusion too, much more would have to be said to make it fully convincing; a single, short article can only scratch the surface. Indeed, a whole book would seem to be needed to do an adequate job. The present paper is perhaps best seen as motivating a corresponding book project. I hope I will have a chance to pursue such a project further in the near future.

Bibliography

1. P. Mancosu, K. Jørgensen and S. Pedersen, eds., *Visualization, explanation and reasoning styles in mathematics* (Springer, Dordrecht, 2005).
2. J. Ferreirós and J. Gray, eds., *The architecture of modern mathematics. Essays in history and philosophy* Oxford University Press, Oxford, 2006).
3. B. Van Kerkhove and J. P. Van Bendegem, eds., *Perspectives on mathematical practices. Bringing together philosophy of mathematics, sociology of mathematics, and mathematics education* (Springer, Dordrecht, 2007).
4. P. Mancosu, ed., *The philosophy of mathematical practice* (Oxford University Press, Oxford, 2008).
5. H. Stein, *Logos, logic, and logistiké*. Some philosophical remarks on nineteenth-century transformations of mathematics, in *History and philosophy of modern mathematics. Minnesota studies in the philosophy of science vol. XI*, eds. W. Aspray and P. Kitcher (University of Minnesota Press, Minneapolis, 1988) pp. 238–59.
6. J. Ferreirós, *Labyrinth of thought. A history of set theory and its role in modern mathematics* (Birkhäuser, Boston, 1999).
7. J. Avigad, Methodology and metaphysics in the development of Dedekind's theory of ideals, in *The architecture of modern mathematics. Essays in history and philosophy*, eds. J. Ferreirós and J. Gray (Oxford University Press, Oxford, 2006) pp. 159–186.
8. R. Dedekind, Continuity and irrational numbers (Stetigkeit und Irrationale Zahlen), in *Essays on the theory of numbers*, ed. and transl. W. Beman (Dover, New York, 1963) pp. 1–27. First published in 1872.
9. C. Boyer and U. Merzbach, *A history of mathematics* (Wiley & Sons, New York, 1991). Second (revised) edition.
10. R. Cooke, Richard Dedekind, Stetigkeit und irrationale Zahlen (1872), in *Landmark writings in western mathematics, 1640-1940*, ed. I. Grattan-Guinness (Elsevier, Dordrecht, 2005) pp. 553–63.
11. S. Awodey and E. Reck, Categoricity and completeness, part I. From nineteenth-century axiomatics to twentieth-century metalogic, *History and Philosophy of Logic* 23, 1–30 (2002).
12. R. Dedekind, The nature and meaning of numbers (Was sind und was sollen die Zahlen?), in *Essays on the theory of numbers*, ed. and transl. W. Beman (Dover, New York, 1963) pp. 31–115. First published in 1888.
13. E. Reck, Dedekind's structuralism: An interpretation and partial defense, *Synthese* 137, 369–419 (2003).
14. J. Ferreirós, Richard Dedekind (1888) and Giuseppe Peano (1889), Booklets on the foundations of arithmetic, in *Landmark writings in western mathematics, 1640-1940*, ed. I. Grattan-Guinness (Elsevier, Dordrecht, 2005) pp. 613–26.
15. W. Sieg and D. Schlimm, Dedekind's analysis of number: System and axioms, *Synthese* 147, 121–70 (2005).
16. J. Stillwell, *Mathematics and its history* (Springer, New York, 2000). Second (revised) edition.
17. P. G. L. Dirichlet and R. Dedekind, *Dirichlet's Vorlesungen über Zahlentheorie, with extensive supplements by Dedekind* (Vieweg, Braunschweig, 1894). English translation: *Lectures on number theory*, ed. and transl. J. Stillwell (American Mathematical Society, Providence RI, 1999).
18. R. Dedekind, *Sur la théorie des nombres entiers algébriques* (Gauthier-Villars, Paris, 1877). English translation: *Theory of algebraic integers*, ed. and transl. J. Stillwell (Cambridge University Press, Cambridge, 1996).
19. C. McLarty, Emmy Noether's 'set theoretic' topology: From Dedekind to the rise of functors, in *The architecture of modern mathematics. Essays in history and philosophy*, eds. J. Ferreirós and J. Gray (Oxford University Press, Oxford, 2006) pp. 187–208.
20. H. M. Edwards, The genesis of ideal theory, *Archive for History of Exact Sciences* 23, 321–378 (1980).
21. D. Reed, *Figures of thought. Mathematics and mathematical texts* (Routledge: London, 1995).
22. D. Corfield, *Towards a philosophy of real mathematics* (Cambridge University Press, 2003).
23. H. M. Edwards, *Divisor theory* (Birkhäuser, Boston, 1990).
24. H. M. Edwards, *Essays in constructive mathematics* (Springer, New York, 2005).
25. H. M. Edwards, Dedekind's invention of Ideal Theory, *Bulletin of the London Mathematical Society* 15, 8–17 (1983).
26. I. Grattan-Guinness, ed., *The rainbow of mathematics. A history of the mathematical sciences* (Norton, London, 1997).
27. R. Haubrich, Zur Entstehung der algebraischen Zahlentheorie Richard Dedekinds (Ph.D. dissertation, University of Göttingen, 1992).
28. L. Corry, *Modern algebra and the rise of mathematical structures* (Birkhäuser, Boston, 2004). Second (revised) edition.
29. W. W. Tait, Frege versus Cantor and Dedekind: On the concept of number, in *Frege: Importance and legacy*, ed. M. Schirn (de Gruyter, Berlin, 1996) pp. 70–113.
30. J. Tappenden, The Caesar Problem in its historical context: Mathematical background, *Dialectica* 59, 1–28 (2005).
31. J. Tappenden, The Riemannian background to Frege's philosophy, in *The architecture of modern mathematics. Essays in history and philosophy*, eds. J. Ferreirós and J. Gray (Oxford University Press, Oxford, 2006) pp. 97–132.
32. P. Mancosu, Style in history and philosophy of mathematics: An overview of the literature, unpublished typescript.

33. I. Hacking, Language, truth and reason, in *Rationality and relativism*, eds. M. Hollis and S. Lukes (Blackwell, Oxford, 1982) pp. 48–66.
34. I. Hacking, 'Style' for historians and philosophers, *Studies in History and Philosophy* 23, 1–20 (1992).
35. J. Høyrup, Tertium non datur: On reasoning styles in early mathematics, in *Visualization, explanation and reasoning styles in mathematics*, eds. P. Mancosu, K. Jørgensen and S. Pedersen (Springer, Dordrecht, 2005) pp. 91–121.
36. A. Davidson, Styles of reasoning: From the history of art to the epistemology of science, in *The emergence of sexuality. Historical epistemology and the formation of concepts* (Harvard University Press, Cambridge, 2001) pp. 125–41.
37. W. Salmon, *Scientific explanation and the causal structure of the world* (Princeton University Press, Princeton, 1984).
38. B. Van Fraassen, *The scientific image* (Clarendon Press, Oxford, 1980).
39. L. Wright, Causal explanations, unpublished typescript.
40. M. Steiner, Mathematical explanation, *Philosophical Studies* 34, 135–51 (1978).
41. P. Kitcher, Explanatory unification, *Philosophy of Science* 48, 507–31 (1981).
42. J. Hafner and P. Mancosu, The varieties of mathematical explanation, in *Visualization, explanation and reasoning styles in mathematics*, eds. P. Mancosu, K. Jørgensen and S. Pedersen (Springer, Dordrecht, 2005) pp. 215–50.
43. M. Hallett, Towards a theory of mathematical research programmes, (I) and (II), *British Journal for the Philosophy of Science* 30, 1–25 and 135–59 (1979).
44. J. Avigad, Mathematical method and proof, *Synthese* 153, 105–59 (2006).
45. K. Vervloesem, Mathematical concepts in computer proofs, to appear in *Philosophical perspectives on mathematical practice*, eds. B. Van Kerkhove, R. Desmet and J. P. Van Bendegem (College Publications, London).
46. J. Tappenden, Proof style and understanding in mathematics I: Visualization, unification, and axiom choice, in *Visualization, explanation and reasoning styles in mathematics*, eds. P. Mancosu, K. Jørgensen and S. Pedersen (Springer, Dordrecht, 2005) pp. 147–214.
47. H. Stein, The enterprise of understanding and the enterprise of knowledge, *Synthese* 140, 135–76 (2004).

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